Assignment 7.

1. $[x \ln 2x - x]_1^a = a \ln 2a - a - \ln 2 + 1$

2. (a) $u = \tan x$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x$, then it equals to $\displaystyle\int_0^1 u^n \,\mathrm{d}u = \frac{1}{n}$

   (b) i. $= \displaystyle\int_0^{\frac{1}{4}\pi} \sec^2 x(\sec^2 x - 1)\,\mathrm{d}x = \int_0^{\frac{1}{4}\pi} (1 + \tan^2 x)\tan^2 x \,\mathrm{d}x = \int_0^{\frac{1}{4}\pi} \tan^2 x + \tan^4 x \,\mathrm{d}x = \frac{1}{3}$

   ii. Split into $t^9 + t^7 + 4(t^7 + t^5) + t^5 + t^3$, final answer $\frac{25}{24}$

3. (a) 0.685

   (b) $\frac{8}{15}$

4. $\ln(\frac{16}{9})$

Assignment 8.

1. (a) Omit

   (b) $\dfrac{10u}{(3-u)(2+u)} = \dfrac{6}{3-u} + \dfrac{-4}{2+u}$

2. (a) $f(x) = \dfrac{3}{3x+2} + \dfrac{-x+3}{x^2+4}$

   (b) $\dfrac{3}{2}\ln 2 + \dfrac{3}{8}\pi.$

3. (a) $y = x - 1$

   (b) $\dfrac{1}{4}(\mathrm{e}^2 - 1)\pi$

4. $\dfrac{x}{\sqrt{x^2+1}} \ln x - \ln\left|x + \sqrt{1+x^2}\right| + C.$

Assignment 7.

1. It is given that $\int_1^a \ln(2x)\,\mathrm{d}x = 1$, where $a > 1$.

   Show that $a = \frac{1}{2}\exp\left(1 + \frac{\ln 2}{a}\right)$, where $\exp(x)$ denotes e^x. [6]

$u = \ln 2x \quad v' = 1$

$u' = \frac{1}{x} \quad v = x$

$\left[x\ln 2x - x\right]_1^a = (a\ln 2a - a) - (\ln 2 - 1) = 1$

$a\ln 2a - a - \ln 2 = 0$

$a\ln 2a = a + \ln 2 \Rightarrow \ln 2a = \frac{a + \ln 2}{a}$

$= 1 + \frac{\ln 2}{a}$

$\Rightarrow a = \frac{\exp\left(1 + \frac{\ln 2}{a}\right)}{2}$

2. (a) Use the substitution $u = \tan x$ to show that, for $n \neq -1$, [4]

$$\int_0^{\frac{1}{4}\pi} \left(\tan^{n+2} x + \tan^n x\right)\,\mathrm{d}x = \frac{1}{n+1}.$$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$\int_0^{\frac{\pi}{4}} \tan^n x \cdot du$

$= \int_0^1 u^n\,du = \left[\frac{1}{1+n} u^{1+n}\right]_0^1 = \frac{1}{n+1}$

(b) Hence find the exact value of

i. $\int_0^{\frac{1}{4}\pi} \left(\sec^4 x - \sec^2 x\right)\,\mathrm{d}x$, [3]

$= \int_0^{\frac{\pi}{4}} \sec^2 x\,(\sec^2 x - 1)\,dx = \int_0^{\frac{\pi}{4}} (1 + \tan^2 x)\tan^2 x\,dx$

$= \int_0^{\frac{\pi}{4}} \tan^2 x + \tan^4 x\,dx$

$= \frac{1}{2+1} = \frac{1}{3}$.

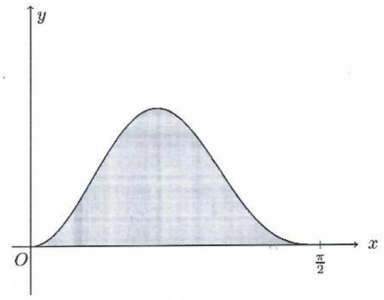
ii. $\int_0^{\frac{1}{4}\pi} \left(\tan^9 x + 5\tan^7 x + 5\tan^5 x + \tan^3 x\right)\,\mathrm{d}x$. [3]

$t^9 + t^7 + 4(t^7 + t^5) + t^5 + t^3$

$\frac{1}{7+1} + 4 \times \frac{1}{5+1} + \frac{1}{3+1}$

$= \frac{1}{8} + \frac{4}{6} + \frac{1}{4} = \frac{3+16+6}{24} = \frac{25}{24}$

3. The diagram shows the curve $y = \sin^2 2x \cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.



(a) Find the x-coordinate of M. [6]

$$\frac{dy}{dx} = 2 \cdot \sin 2x \cdot \cos 2x \cdot 2 \cdot \cos x + \sin^2 2x(-\sin x)$$

$$= \sin 2x \, (4\cos 2x \cos x - \sin 2x \sin x)$$

$$\frac{dy}{dx} = 0 \Rightarrow \sin x \sin 2x = 4\cos x \cos 2x \Rightarrow \tan x \tan 2x = 4$$

Let $t = \tan x$

$$t \cdot \frac{2t}{1-t^2} = 4$$

$$\Rightarrow t = \sqrt{\frac{4}{6}}$$

$$\Rightarrow \tan x = \sqrt{\frac{4}{6}}$$

$$\Rightarrow x = 0.685$$

(b) Using the substitution $u = \sin x$, find by integration the area of the shaded region bounded by the curve and the x-axis. [4]

$u = \sin x$, $\frac{du}{dx} = \cos x$

$x = 0, u = 0$; $x = \frac{\pi}{2}, u = 1$

$$\int_0^{\frac{\pi}{2}} \sin^2 2x \cos x \, dx$$

$$= \int_0^1 4\sin^2 x \cos^2 x \, du$$

$$= \int_0^1 4u^2(1-u^2)\, du = 4\left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1$$

$$= \frac{8}{15}$$

4. (†) Evaluate the integral $\displaystyle\int_{-1}^{1} \left|\frac{x}{x+2}\right| dx$ [7]

$x(x+2) < 0$

$-2 < x < 0$

$$= -\int_{-1}^{0} \frac{x}{x+2} dx + \int_0^1 \frac{x}{x+2} dx$$

$$= \Big[x - 2\ln|x+2|\Big]_0^{-1} + \Big[x - 2\ln|x+2|\Big]_0^1 \,!$$

Total mark of this assignment: $26 + 7$.

The symbol (†) indicates a bonus question. Finish other questions before working on this one.

$$= 2\ln 2 - 1 + 2\ln 2 + 1 - 2\ln 3$$

$$= \ln\frac{16}{9}$$

Assignment 8.

1. Let $I = \int_2^5 \frac{5}{x+\sqrt{6-x}}\,dx$.

   (a) Using the substitution $u = \sqrt{6-x}$, show that [4]

$x=2,\ u=2$ $\qquad u^2 = 6-x$

$x=5,\ u=1$ $\qquad x = 6-u^2$

$$I = \int_1^2 \frac{10u}{(3-u)(2+u)}\,du.$$

$\frac{dx}{du} = -2u$

$I = \int_2^1 \frac{5}{6-u^2+u} \cdot -2u\,du = \int_1^2 \frac{10u}{(3-u)(2+u)}\,du$

   (b) Hence show that $I = 2\ln\left(\frac{9}{2}\right)$. [6]

$\frac{10u}{(3-u)(2+u)} = \frac{A}{3-u} + \frac{B}{2+u}$

$2A+3B=0$

$A-B=10$

$\Rightarrow B=-4.\ A=6.$

$\int_1^2 \left(\frac{6}{3-u} + \frac{-4}{2+u}\right) du$

$= \left[-6\ln|3-u| - 4\ln|2+u|\right]_1^2$

$= [-6\ln 1 - 4\ln 4] - [-6\ln 2 - 4\ln 3]$

$= 6\ln 2 - 8\ln 2 + 4\ln 3$

$= 4\ln 3 - 2\ln 2$

$= 2(2\ln 3 - \ln 2)$

$= 2\ln\left(\frac{9}{2}\right)$

2. Let $f(x) = \frac{7x+18}{(3x+2)(x^2+4)}$.

   (a) Express $f(x)$ in partial fractions. [5]

$\frac{7x+18}{(3x+2)(x^2+4)} = \frac{A}{3x+2} + \frac{Bx+C}{x^2+4}$

$7x+18 = A(x^2+4) + (Bx+C)(3x+2)$

$(A+3B)x^2$ $\qquad B=-1$

$(2B+3C)x$ $\qquad A=3$

$2C+4A$ $\qquad C=3$

   (b) Hence find the exact value of $\int_0^2 f(x)\,dx$. [6]

$A+3B=0$ $\qquad A=-3B$

$2B+3C=7$

$2C+4A=18$

~~$-6B+3C=$~~

~~$2C-12B=18$~~

$3C-18B=27$

$20B=-20$

$B=-1$

$\int_0^2 \frac{3}{3x+2} + \frac{-x+3}{x^2+4}\,dx$

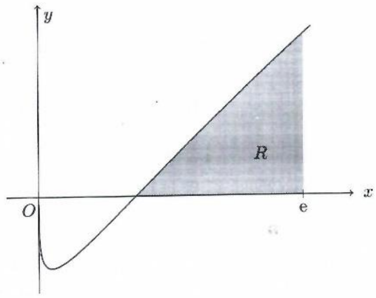
$= \left[\ln|3x+2| - \frac{1}{2}\ln|x^2+4| + \frac{3}{4}\cdot 2\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$

$= \left[\ln 8 - \frac{1}{2}\ln 8 + \frac{3}{2}\tan^{-1}(1)\right] - \left[\ln 2 - \frac{1}{2}\ln 4\right]$

$= \left[\frac{3}{2}\ln 2 + \frac{3}{2}\times\frac{\pi}{4}\right] - 0$

$= \frac{3}{2}\ln 2 + \frac{3}{8}\pi$

3. The diagram shows the curve $y = x^{\frac{1}{2}} \ln x$. The shaded region between the curve, the x–axis and the line $x = \mathrm{e}$ is denoted by R.



(a) Find the equation of the tangent to the curve at the point where $x = 1$, giving your answer in the form $y = mx + c$. [4]

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}\cdot \ln x + x^{\frac{1}{2}}\cdot\frac{1}{x} \qquad x=1,\ y=0$$

$$x=1,\quad \frac{dy}{dx} = \frac{1}{2}\cdot\ln 1 + 1 = 1$$

$$y = x-1$$

(b) Find by integration the volume of the solid obtained when the region R is rotated completely about the x–axis. Give your answer in terms of π and e. [7]

$$\int_1^e \pi y^2\,dx = \int_1^e \pi x \ln^2 x\,dx$$

$$u = \ln^2 x,\quad v' = x$$

$$u' = 2\ln x\cdot\frac{1}{x}\quad v = \frac{1}{2}x^2$$

$$\frac{1}{2}x^2\ln^2 x - \int_1^e x\ln x\,dx$$

~~[8]~~

$$u = \ln x \quad v' = x$$

$$u' = \frac{1}{x}\quad v = \frac{1}{2}x^2$$

$$\pi\left[\frac{1}{2}x^2\ln^2 x - \frac{1}{2}x^2\ln x + \frac{1}{2}\cdot\frac{1}{2}x^2\right]_1^e$$

$$= \left[\frac{1}{2}e^2 - \frac{1}{2}e^2 + \frac{1}{4}e^2\right] - \left[0 - 0 + \frac{1}{4}\right)$$

$$\frac{1}{4}(e^2-1)\pi$$

4. (†) $\displaystyle\int \frac{\ln x\,\mathrm{d}x}{(1+x^2)^{\frac{3}{2}}}$

$$x = \tan\theta \quad \frac{dx}{d\theta} = \sec^2\theta$$

$\sqrt{1+x^2}$, x, 1, θ

$$\int \ln\tan\theta\cdot\frac{1}{\sec^3\theta}\cdot\sec^2\theta\,d\theta$$

$$= \int(\ln\tan\theta)\cdot\cos\theta\,d\theta$$

$$u = \ln\tan\theta \quad v' = \cos\theta$$

$$u' = \frac{1}{\tan\theta}\cdot\sec^2\theta = \frac{1}{\sin\theta\cos\theta}\quad v = \sin\theta$$

Total mark of this assignment: $32 + 8$.

The symbol (†) indicates a bonus question. Finish other questions before working on this one.

$$= \left[\sin\theta\cdot\ln\tan\theta\right) - \int\frac{1}{\cos\theta}\,d\theta$$

$$= \sin\theta\ln\tan\theta - \frac{1}{2}\left(-\ln|1-\sin\theta| + \ln|1+\sin\theta|\right) + C$$

$$= \frac{x}{\sqrt{1+x^2}}\ln x - \frac{1}{2}\left(\ln\frac{x+\sqrt{1+x^2}}{x-\sqrt{1+x^2}}\right) + C$$